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NBSIR 76-993

The Calibration of a Pentaprism

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National Bureau of Standards
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U.S. DEPARTMENT OF COMMERCE, Rogers C. B. Morton, *Secretary*
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THE CALIBRATION OF A PENTAPRISM

by

Charles P. Reeve and Ralph C. Veale

1. Introduction

The Dimensional Technology Section is currently responsible for the calibration of pentaprisms at the National Bureau of Standards. This responsibility requires that we be familiar with

1) a good method of calibrating pentaprisms

2) their common uses

3) effects of errors in their internal geometry, and

4) effects of errors in their alignment with a beam of light.

Information which we have been able to obtain on the above subjects is currently quite scattered throughout different sources and needs to be unified. Additionally, through investigation carried out at the Bureau, we have learned some important facts relative to the functioning of a pentaprism that we have been unable to find documented anywhere else. Therefore, the purpose of this paper is to collect from these sources all the facts about pentaprisms which we believe are relevant to their calibration and usage. These facts are discussed individually in the following sections where material is liberally incorporated from the references.

Although some of the mathematics may appear awesome, the results are fairly simple. The method of calibration is also simple and the data reduction can easily be done by hand with the aid of a small calculator.

2. Description and Uses

The word pentaprism is a commonly used contraction of "pentagonal prism" which, as its name indicates, is a five-sided prism (fig. 1). It is one of a class of objects known as "constant deviation prisms" whose purpose is to bend a beam of light exactly 90° regardless of the

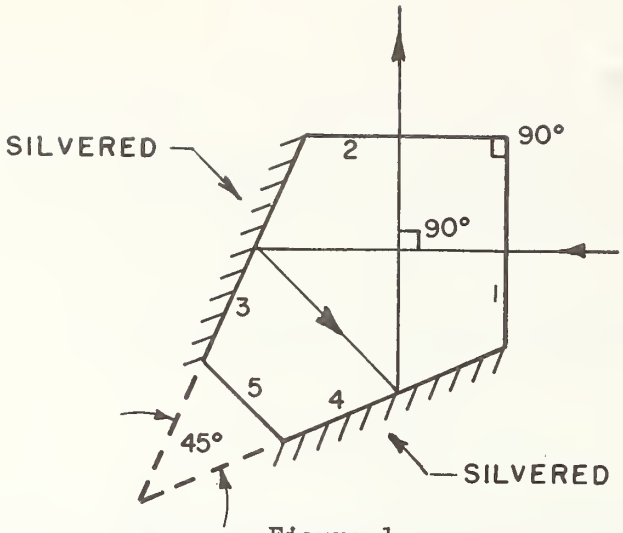


Figure 1

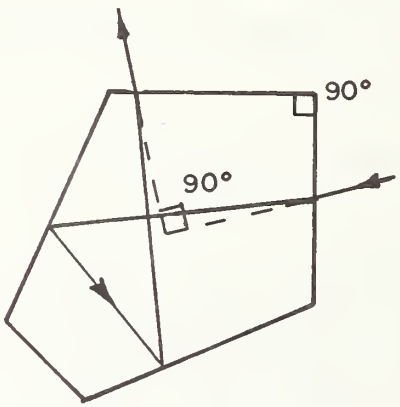


Figure 2

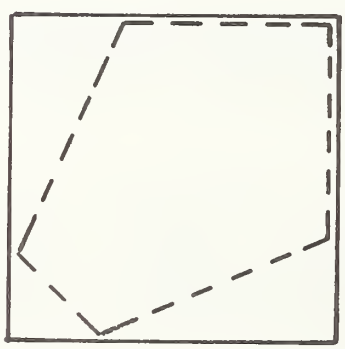


Figure 3

angle of incidence of the light. Each of the five faces is perpendicular to both the base and the top. Faces 1 and 2 serve as entry and exit planes for a beam of light, and the angle between them is nominally 90°. Faces 3 and 4 are silvered and serve as reflectors, and the angle between them is nominally 45°. Face 5 serves no purpose other than truncating the unusable portion of the wedge between faces 3 and 4.

The function of the pentaprism is governed by the law of reflection which states that if an incident beam of light is reflected off two planes, then the reflected beam is bent by an angle which is equal to twice the supplement of the acute angle between the reflectors. Thus in a pentaprism the beam deviation is $2(180 - 45)° = 270°$. It can easily be shown that if the incident beam is not normal to the face which it enters, it will be refracted on entry and exit but through the same angle so that the effects cancel each other (fig. 2). Of course this assumes that the entry and exit faces are exactly 90° apart. The case where that is not true is covered in the next section.

Sometimes a pentaprism is mounted in a square metal housing with ground locating faces and is called an optical square (fig. 3), referring to its functional use. The faces of the pentaprism are usually mounted not quite parallel to the corresponding faces of the mount to avoid ghost reflections directly off the prism face when the mount is aligned to the line of sight. This is especially important if the optical square is being used with an autocollimator because the ghost image could interfere with the returning beam.

A pentaprism (optical square) is useful in many types of optical measurements where the line of sight or a beam of light must be turned at a right angle. For example, it may be used with an autocollimator to check the parallelism of internal surfaces, perpendicularity of a translational movement, or relative squareness of two or more machine surfaces. These and other examples are given in Farago [1]* and Hume [2,3].

3. Geometrical Models

In the past it was usually assumed that if a pentaprism was observed to bend a beam of light within a few seconds of a right angle then the 45° and 90° angles must also be within a few seconds of nominal. This has been found to be an erroneous assumption. A recent sampling of five pentaprisms indicated that the 90° angle is likely to deviate several minutes from nominal. In view of this it is mandatory to examine the effects of such deviations. This is accomplished in the next two sections. Also considered will be two characteristics of a "per-

*Numbers in brackets indicate references listed at the end of this paper.

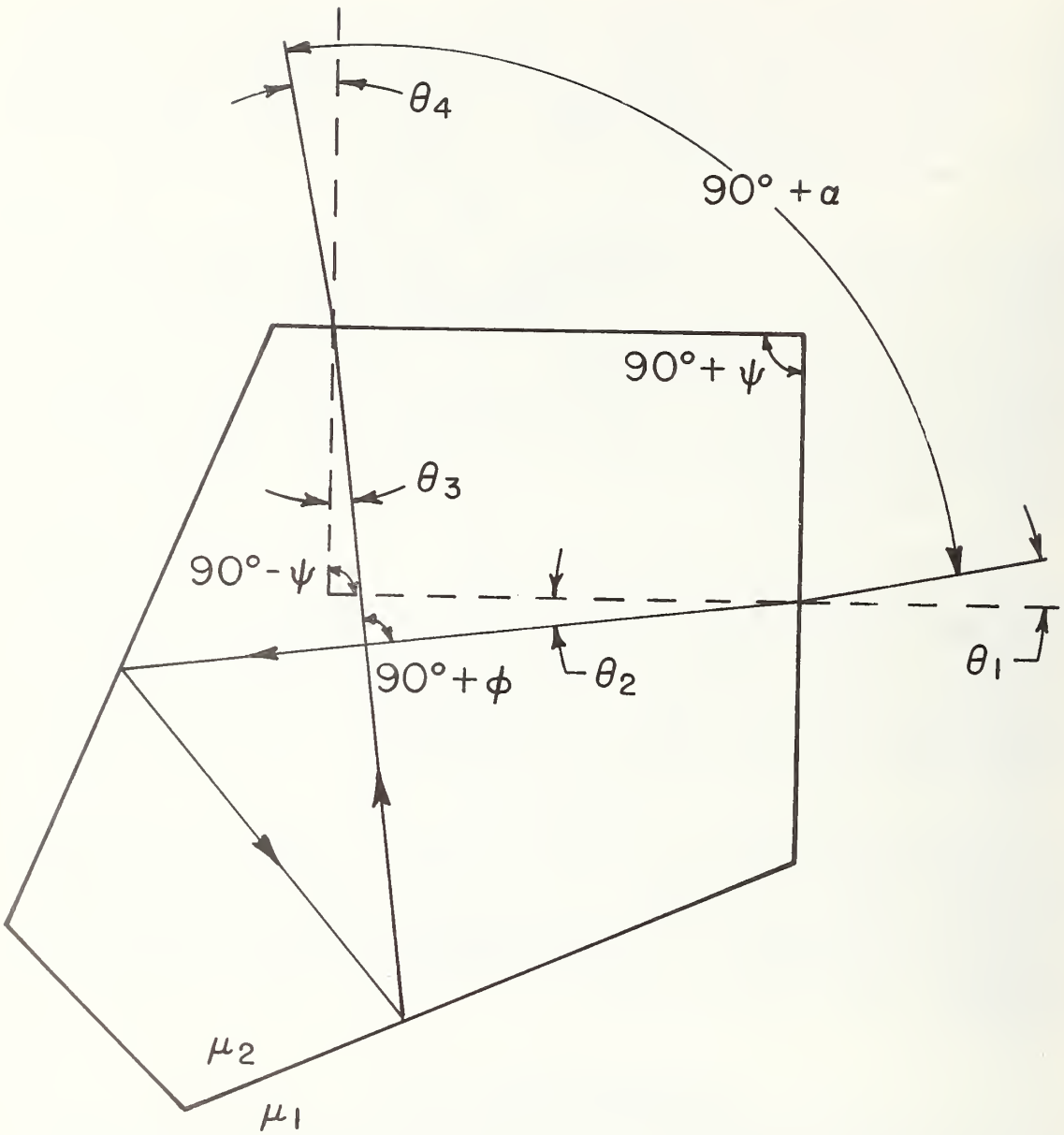


Figure 4

fect" pentaprism: the effects of tilting it in a vertical plane and its usable angular range of incident light.

3.1. Angular Errors between Faces - Transmitted Beam

Assume here that the 90° angle between the faces is in error by a small amount ψ and that the internal bending of the beam is in error by a small amount ϕ due to an error of $\frac{\phi}{2}$ in the 45° wedge angle. The path of a transmitted beam in a horizontal plane is shown in figure 4. Let the observed bending angle of the beam outside the pentaprism be given by $90° + \alpha$. It is then essential to know α as a function of ϕ, ψ, θ_1, μ_1, and μ_2 where θ_1 is the deviation of the incident beam from normal, μ_1 is the index of refraction of the medium (usually air), and μ_2 is the index of refraction of the pentaprism. Assume all angles in figure 4 are positive as shown. Then $90° - \psi + \theta_3 - \theta_2 = 90° + \phi$, or

$$\phi + \psi = \theta_3 - \theta_2, \tag{3-1}$$

and $90° - \psi + \theta_4 - \theta_1 = 90° + \alpha$, or

$$\alpha + \psi = \theta_4 - \theta_1. \tag{3-2}$$

The refraction law gives

$$\begin{aligned} \mu_1 \sin \theta_1 &= \mu_2 \sin \theta_2, \text{ and} \\ \mu_1 \sin \theta_4 &= \mu_2 \sin \theta_3. \end{aligned} \tag{3-3}$$

Writing these equations in the form

$$\begin{aligned} \alpha &= \theta_4 - \theta_1 - \psi, \\ \theta_4 &= \mathrm{Sin}^{-1}\left(\frac{\mu_2}{\mu_1} \sin \theta_3\right), \\ \theta_3 &= \phi + \psi + \theta_2, \\ \theta_2 &= \mathrm{Sin}^{-1}\left(\frac{\mu_1}{\mu_2} \sin \theta_1\right), \end{aligned} \tag{3-4}$$

and substituting from bottom to top gives the expression

$$\alpha = \mathrm{Sin}^{-1}\left\{\frac{\mu_2}{\mu_1}\sin\left[\phi + \psi + \mathrm{Sin}^{-1}\left(\frac{\mu_1}{\mu_2}\sin\theta_1\right)\right]\right\} - \theta_1 - \psi. \qquad (3\text{-}5)$$

Using the power series expansions for sine and arcsine functions and ignoring all powers of θ_1 greater than two and powers of $\phi + \psi$ greater than one, equation (3-5) reduces to the approximation equation

$$\alpha \approx \frac{\mu_2}{\mu_1}\phi + \left(\frac{\mu_2}{\mu_1} - 1\right)\psi + \frac{1}{2}\left(\frac{\mu_2}{\mu_1} - \frac{\mu_1}{\mu_2}\right)(\phi + \psi)\,\theta_1^2. \qquad (3\text{-}6)$$

Let $\alpha(\phi, \psi, \theta_1, \mu_1, \mu_2)$ be the representation of α function of the five variables in the system, and note that

$$\alpha(0, 0, \theta_1, \mu_1, \mu_2) = 0 \qquad (3\text{-}7)$$

in both the true (3-5) and approximate (3-6) expressions for α. This indicates that a well-made pentaprism is a true square for all normal and off-normal beams.

Let $\Delta\alpha$ be defined by

$$\Delta\alpha = \alpha(\phi, \psi, \theta_1, \mu_1, \mu_2) - \alpha(\phi, \psi, 0, \mu_1, \mu_2). \qquad (3\text{-}8)$$

Then $\Delta\alpha$ is the fluctuation of α as θ_1 varies from zero to some maximum value. In equation (3-5) this fluctuation is a maximum when ϕ, ψ, and θ_1 all have the same algebraic sign. In table 1 $\Delta\alpha$ is evaluated using equations (3-5) and (3-6) for specific values of ϕ, ψ, θ_1, μ_1, and μ_2. The values from the approximate expression (3-6)* are in parentheses. It is obvious that the approximate expression is very good over this range of values for the arguments. The effectiveness of the pentaprism, in terms of the small magnitude of $\Delta\alpha$ caused by manufacturing angle errors ϕ and ψ and alignment angle error θ_1, is impressive.

3.2. Angular Errors between Faces - Reflected Beams

Figure 5 illustrates the paths of the two most intense reflected "ghost" beams for the pentaprism. The angle between these beams, $2\xi = \theta_7 - \theta_8$, is a function of ϕ, ψ, θ_1, μ_1, and μ_2. Later it will

*When evaluating this expression angles must be expressed in radians.

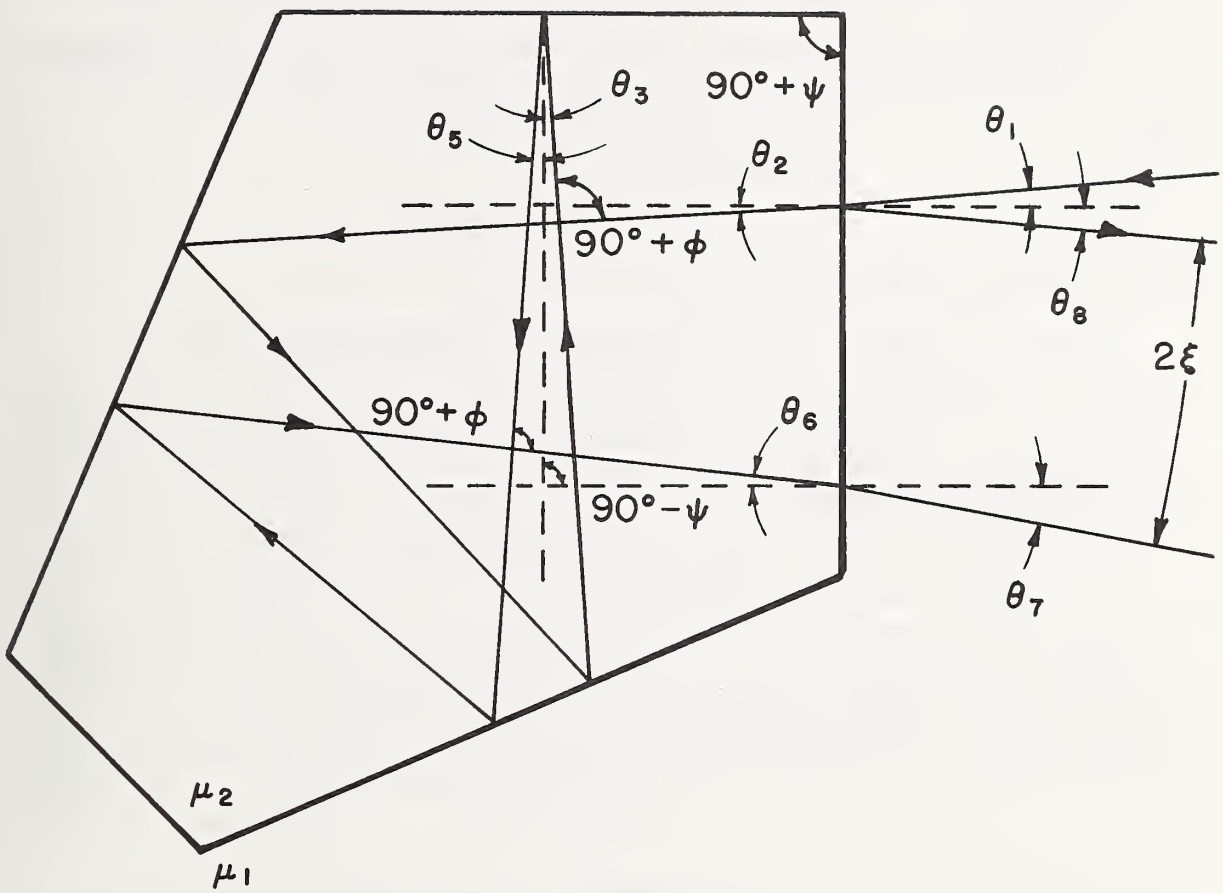

θ_3
$90°+\psi$
θ_5
θ_2
θ_1
$90°+\phi$
θ_8
2ξ
$90°+\phi$
θ_6
$90°-\psi$
θ_7
μ_2
μ_1

Figure 5

be shown how the measurement of 2ξ in conjunction with the measurement of α can be used to calibrate the interior angles of the penta-prism.

Assume that all angles in figure 5 are positive as shown. Then

$$90° - \psi + \theta_3 - \theta_2 = 90° + \phi, \text{ or}$$

$$\phi + \psi = \theta_3 - \theta_2, \tag{3-9}$$

and

$$90° - \psi - \theta_5 + \theta_6 = 90° + \phi, \text{ or}$$

$$\phi + \psi = \theta_6 - \theta_5. \tag{3-10}$$

The refraction law gives

$$\mu_1 \sin \theta_1 = \mu_2 \sin \theta_2 \text{ and}$$

$$\mu_1 \sin \theta_7 = \mu_2 \sin \theta_6, \tag{3-11}$$

and the reflection law gives

$$\theta_5 = \theta_3 \text{ and}$$

$$\theta_8 = \theta_1. \tag{3-12}$$

Writing these equations in the form

$$2\xi = \theta_7 - \theta_8,$$

$$\theta_8 = \theta_1,$$

$$\theta_7 = \text{Sin}^{-1}\left(\frac{\mu_2}{\mu_1} \sin \theta_6\right),$$

$$\theta_6 = \phi + \psi + \theta_5, \tag{3-13}$$

$$\theta_5 = \theta_3,$$

$$\theta_3 = \phi + \psi + \theta_2,$$

$$\theta_2 = \text{Sin}^{-1}\left(\frac{\mu_1}{\mu_2} \sin \theta_1\right),$$

and substituting from bottom to top gives the expression

$$2\xi = \mathrm{Sin}^{-1}\left\{\frac{\mu_2}{\mu_1}\sin\left[2(\phi+\psi) + \mathrm{Sin}^{-1}\left(\frac{\mu_1}{\mu_2}\sin\theta_1\right)\right]\right\} - \theta_1. \qquad (3\text{-}14)$$

When 2ξ is measured with an autocollimator the pentaprism is of necessity adjusted so that θ_1 is small (usually less than 5'). Therefore equation (3-14) can be reduced using the first order approximations to the sine and arcsine functions to give the approximation equation

$$2\xi = 2\frac{\mu_2}{\mu_1}(\phi+\psi). \qquad (3\text{-}15)$$

3.3. Errors in Vertical Alignment with Incident Beam

Now consider the case of a perfect pentaprism where the incident beam is not perpendicular to the pentaprism face in a vertical plane. Assume that the incident beam is horizontal but the pentaprism is tilted toward the beam by an angle τ as shown in figures 6 and 8. It is natural to assume that the transmitted beam would show both a vertical and horizontal component of change from the $\tau = 0$ position (fig. 7). An experiment was conducted where a pentaprism was tilted through a 9' arc with a sine bar and the vertical and horizontal changes in angle were read by an autocollimator. The observed data are given in table 2. There is apparently a one-to-one correspondence between the tilt angle and the vertical deflection of the beam, but no correspondence between the tilt angle and the horizontal deflection of the beam. The vertical deflection may be of interest in certain situations, but not in the calibration process since the autocollimators are set to read horizontal angle only. Therefore consideration will be given only to deflection in the horizontal plane.

It is interesting to verify geometrically that the tilt angle does not significantly affect the angle of deflection horizontally. When the pentaprism is in a horizontal position with $\tau=0$ as shown in figure 7, consider the points a, b, and c which lie at the intersection of the plane z=h and the three vertical edges which form the 45° wedge angle. (Note that the point c lies on the truncated edge.) Let the pentaprism be tilted by rotating it about the x axis through an angle τ, and consider the new points a', b', and c' which lie at the intersection of the plane z=h and the same three vertical edges respectively as shown in figures 6 and 8. By calculating the (x,y) coordinates of the six points the apparent change in the horizontal wedge angle γ can be computed. The coordinates are as follows:

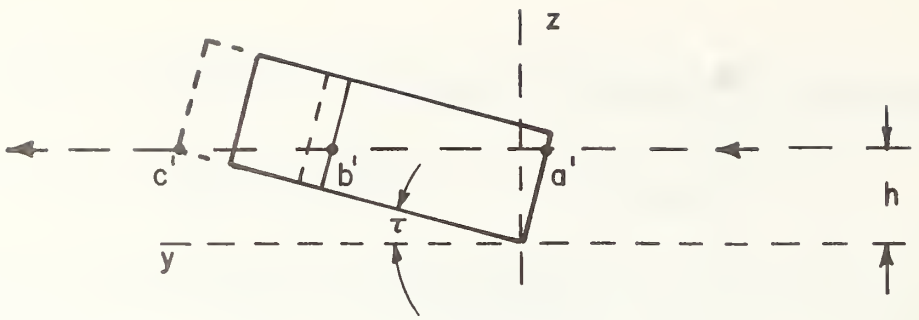


Figure 6

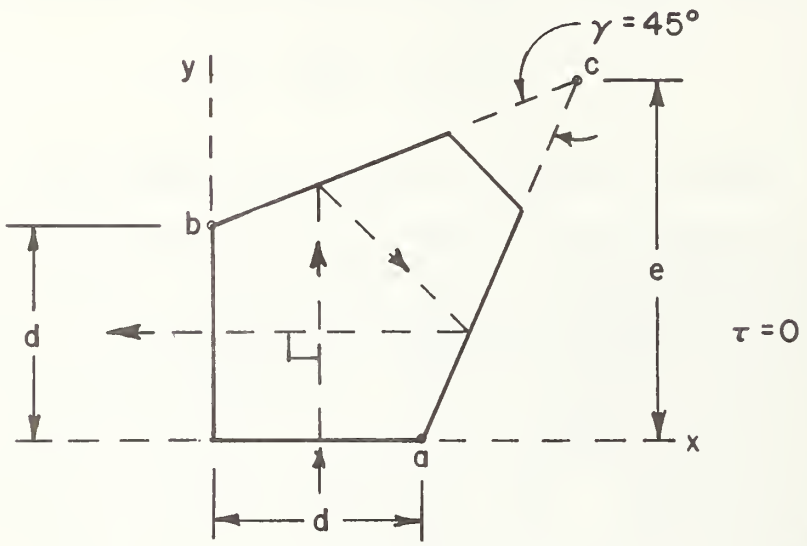


Figure 7

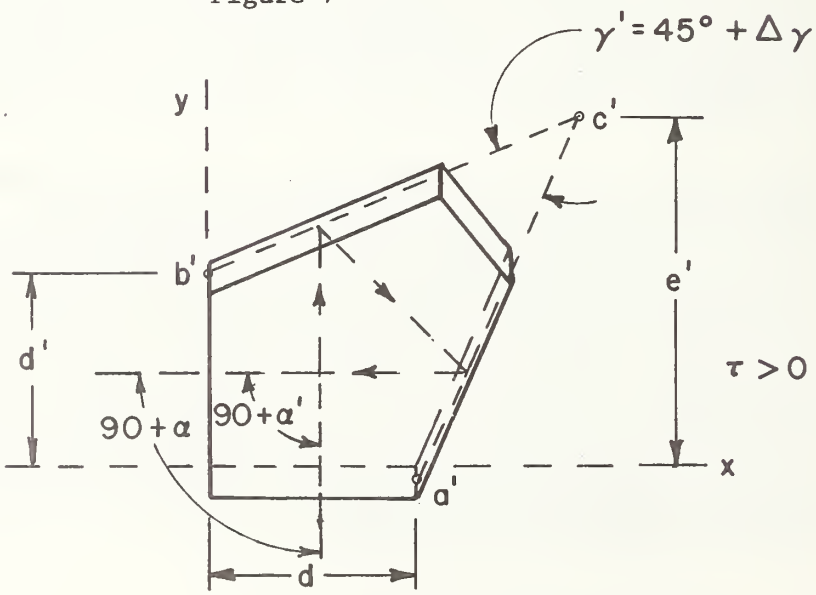


Figure 8

point	(x,y)	point	(x,y)
a	(d,0)	a'	$(d, \frac{-e \sin^2\tau}{\cos \tau})$
b	(0,d)	b'	$(0, \frac{d}{\cos \tau} - \frac{e \sin^2\tau}{\cos \tau})$
c	(e,e)	c'	(e, e cos τ).

If the origin is shifted to point c' then the coordinates of a' and b' become

$$(d-e, \frac{-e \sin^2\tau}{\cos \tau} - e \cos \tau) = (d-e, \frac{-e}{\cos \tau}) \text{ and}$$

$$(-e, \frac{d}{\cos \tau} - \frac{e \sin^2\tau}{\cos \tau} - e \cos \tau) = (-e, \frac{d-e}{\cos \tau})$$

respectively. Consider these two points to be vectors $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the (x,y) plane and recall that the expression for their dot product can be written

$$\cos \gamma' = \frac{A \cdot B}{|A||B|} \tag{3-16}$$

where $A \cdot B = a_1 b_1 + a_2 b_2$, $|A| = \sqrt{a_1^2 + a_2^2}$, $|B| = \sqrt{b_1^2 + b_2^2}$, and γ' is the angle between the vectors. Then

$$\cos \gamma' = \frac{e(e-d) + \frac{e(e-d)}{\cos^2\tau}}{\sqrt{\left[(d-e)^2 + \frac{e^2}{\cos^2\tau}\right]\left[e^2 + \frac{(d-e)^2}{\cos^2\tau}\right]}} \quad . \tag{3-17}$$

Substituting $d = e(1 - \tan 22\ 1/2°) = e(2 - \sqrt{2})$ into eq. (3-17) gives

$$\cos \gamma' = \frac{\cos^2\tau + 1}{\sqrt{\cos^4\tau + 6 \cos^2\tau + 1}} \tag{3-18}$$

Let $\gamma' = \gamma + \Delta\gamma = 45° + \Delta\gamma$. Then

$$\Delta\gamma = \mathrm{Cos}^{-1}\left[\frac{\cos^2\tau + 1}{\sqrt{\cos^4\tau + 6\cos^2\tau + 1}}\right] - 45^\circ \ . \quad (3\text{-}19)$$

Now consider the effect of $\Delta\gamma$ on the deviation α' of the reflected beam from a right angle inside the prism (fig. 8). The reflection law gives $270^\circ + \alpha' = 2(180^\circ - \gamma') = 2(180^\circ - 45^\circ - \Delta\gamma) = 270^\circ - 2\Delta\gamma$, hence

$$\alpha' = -2\Delta\gamma. \quad (3\text{-}20)$$

Since the exiting beam is not perpendicular to the prism face it is refracted. The resulting deviation α of the reflected beam from a right angle outside the prism is related to α' by

$$\alpha = \mathrm{Sin}^{-1}\left(\frac{\mu_2}{\mu_1}\sin\alpha'\right), \quad (3\text{-}21)$$

hence

$$\alpha = \mathrm{Sin}^{-1}\left[\frac{\mu_2}{\mu_1}\sin(-2\Delta\gamma)\right]. \quad (3\text{-}22)$$

Substituting eq. (3-19) into eq. (3-22) gives

$$\alpha = \mathrm{Sin}^{-1}\left[\frac{\mu_2(\cos^4\tau - 2\cos^2\tau + 1)}{\mu_1(\cos^4\tau + 6\cos^2\tau + 1)}\right]. \quad (3\text{-}23)$$

If τ is small then eq. (3-23) can be approximated by

$$\alpha \approx \left(\frac{\mu_2}{\mu_1}\right)\frac{\tau^4}{8} \ . \quad (3\text{-}24)$$

Values of α are computed for selected values of τ in table 3 using the exact equation (3-23) and the approximate equation (3-24). It is very clear from table 3 that any tilt angle less than 3° has no measurable effect on the reflected beam in the horizontal plane. Since the tilt during calibration is never more than a few minutes, the tilt angle can be dismissed as a cause for concern.

It should be noted that a tilt along the base of the face which is perpendicular to the one just considered gives the same effect since the pentaprism functions in a symmetric manner.

3.4. Maximum Deviation of Incident Beam from Normal

Occasionally it may be of interest to know the range of incident beam angles for which there is transmission through the pentaprism. The two extreme cases (for a perfect pentaprism) are shown in figure 9 where θ_i is the deviation of the incident beam from normal and θ_r is the deviation of the refracted beam from normal. Some mathematical manipulation gives the result

$$\theta_r = \mathrm{Tan}^{-1}\left(\frac{1}{2+\sqrt{2}}\right) = 16^\circ\ 19'\ 29.8'', \tag{3-25}$$

hence

$$\theta_i = \mathrm{Sin}^{-1}\left(\frac{\mu_2}{\mu_1}\sin\theta_r\right) = \mathrm{Sin}^{-1}\left(\frac{\mu_2}{\mu_1\sqrt{7+4\sqrt{2}}}\right). \tag{3-26}$$

If $\mu_1 = 1.0$ and $\mu_2 = 1.5$, then $\theta_i = 24^\circ\ 56'\ 14.4''$.

For incident beams entering at the center of the face (fig. 10), the maximum deviation is

$$\theta_r = \mathrm{Tan}^{-1}\left(\frac{1}{4+.2\sqrt{2}}\right) = 8^\circ\ 19'\ 53.6'', \tag{3-27}$$

hence

$$\theta_i = \mathrm{Sin}^{-1}\left(\frac{\mu_2}{\mu_1}\sin\theta_r\right) = \mathrm{Sin}^{-1}\left(\frac{\mu_2}{\mu_1\sqrt{25+16\sqrt{2}}}\right). \tag{3-28}$$

If $\mu_1 = 1.0$ and $\mu_2 = 1.5$, then $\theta_i = 12^\circ\ 33'\ 12.7''$.

4. Method of Calibration

The method of calibrating α has been in use at the National Bureau of Standards for several years and has similarities to a method described by Hume [3]. The method of calibrating the remaining parameters was recently developed at NBS and has been implemented as part of the calibration service.

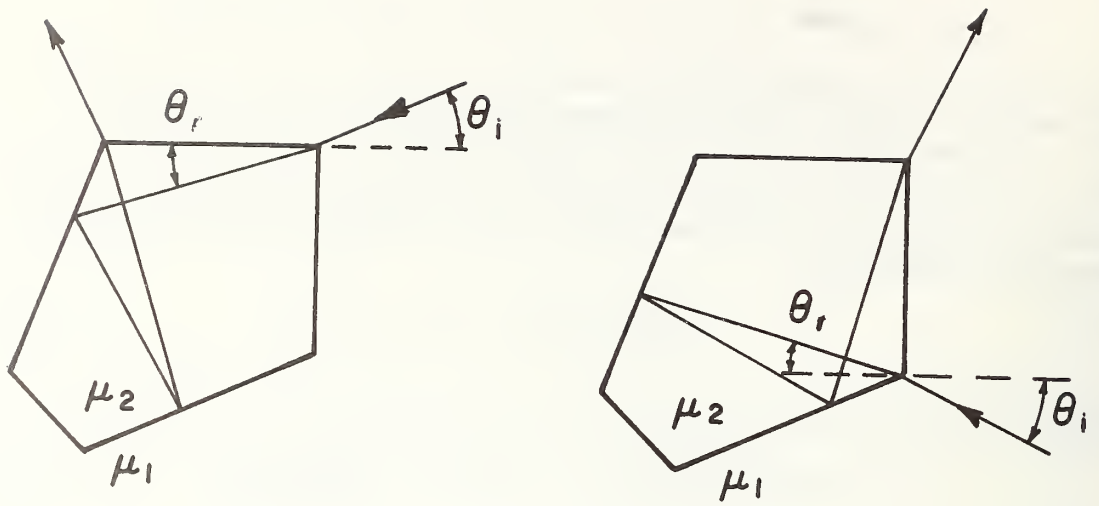


Figure 9

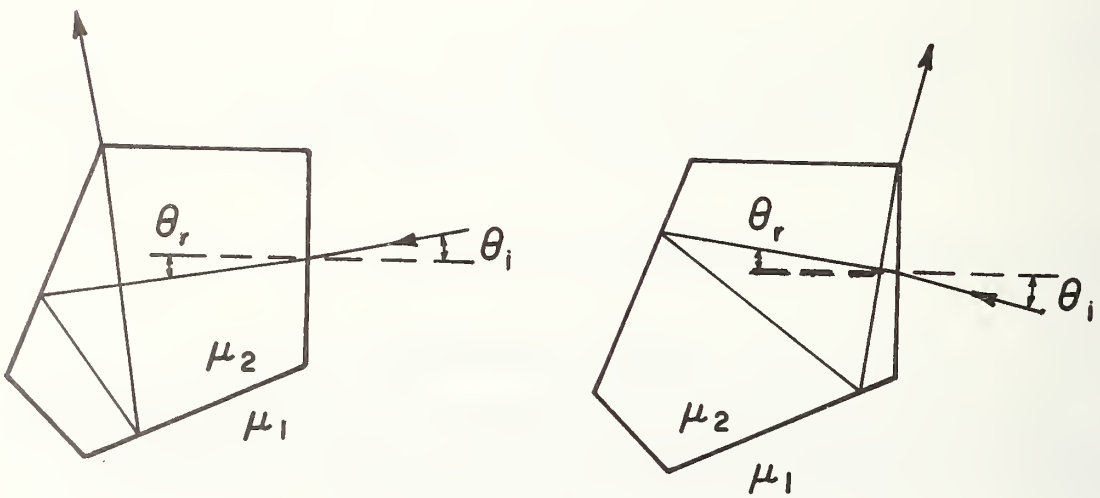


Figure 10

4.1. Setup Procedures

The measurement setup calls for a surface plate, two autocollimators, and two high quality front surface type mirrors. The autocollimators and mirrors are set on the surface plate as shown in fig. 11. The mirrors should be mounted so they are perpendicular to the surface plate. The angles between the axes of the autocollimators and the perpendiculars to the mirrors should be close enough to 90° that either autocollimator will read on scale with any of the four possible orientations of the pentaprism. This can be accomplished by placing the pentaprism so that it reflects the beam from autocollimator A into mirror 1. Mirror 1 is adjusted until autocollimator A reads on scale, then the pentaprism is rotated 90° so that it reflects the beam from autocollimator B into mirror 1. Autocollimator B is adjusted so that it reads on scale, then the pentaprism is rotated another 90° so that it reflects the beam from autocollimator B into mirror 2. Mirror 2 is adjusted so that autocollimator B reads on scale. At this point everything should be adequately aligned, but as a check the pentaprism should be rotated 90° again to be sure that autocollimator A and mirror 2 are properly aligned.

Each autocollimator should be checked for reading vertical angle by tilting the pentaprism slightly toward it and observing a change in reading. If there is no change then the autocollimator is properly reading horizontal angle only. After all adjustments are completed there should follow a period of waiting time for temperature stabilization.

4.2. Measurement of Parameters

In the normal usage of a pentaprism the angle of deflection of the transmitted beam, α, is considered to be the most important parameter. However, in a high precision calibration of a pentaprism it is necessary to know the magnitude of other parameters which may affect α in certain situations. The methods of measuring these parameters are outlined in the following sections.

4.2.1. Deflection Angle of Transmitted Beam

Let the angles between the autocollimator axes and the perpendiculars to the mirrors be given by $90° + \beta_1$, $90° + \beta_2$, $90° + \beta_3$, and $90° + \beta_4$ as shown in figure 11. (The autocollimator axis is taken to be the line along which the collimated light exits.) Readings are made with the pentaprism in each of its four positions. Care should be taken that with each reading the front surface reflection of the pentaprism is outside the field of view of the autocollimator. The readings take the form

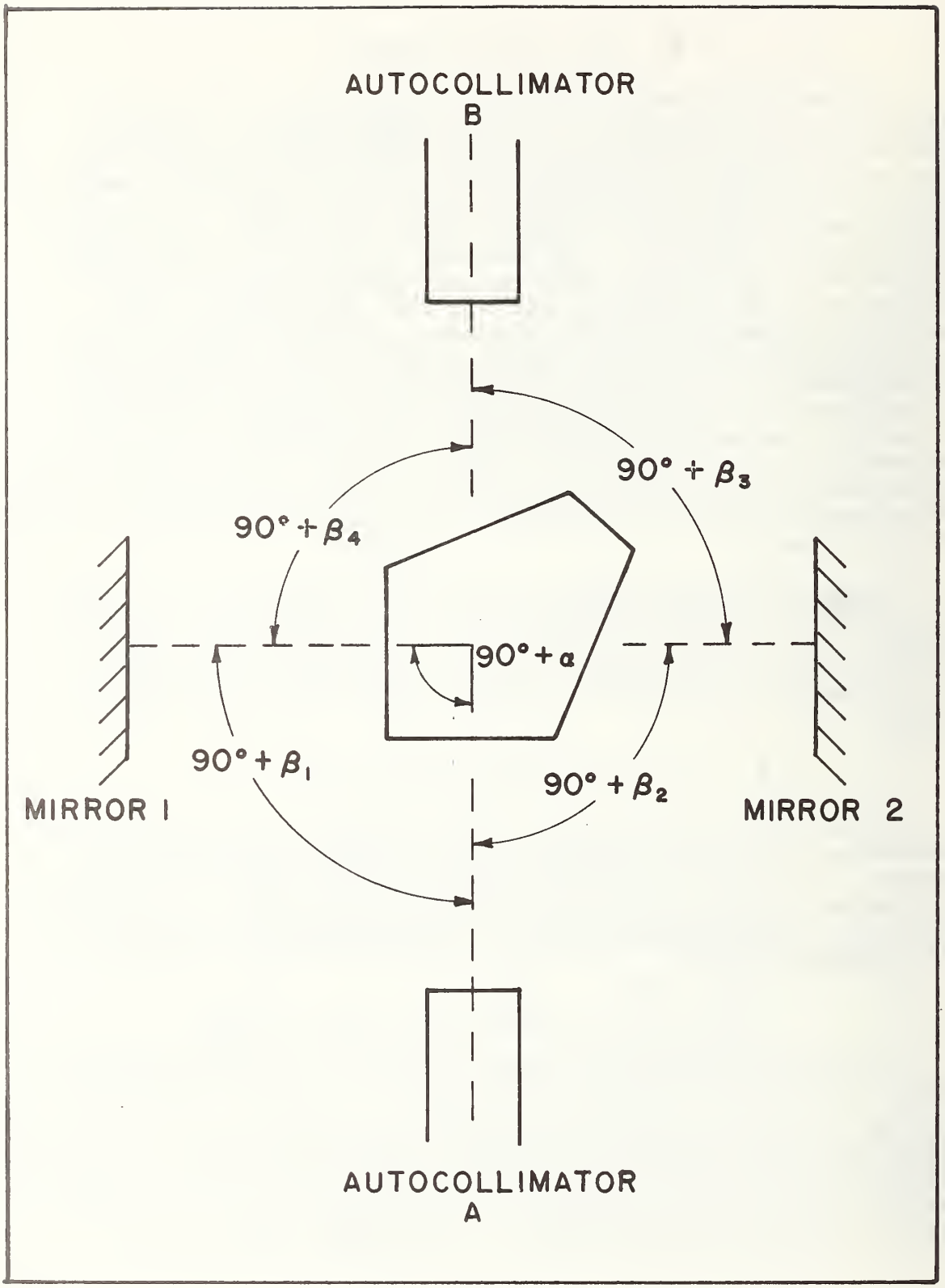

AUTOCOLLIMATOR
B
90° + β3
90° + β4
90° + α
90° + β1
90° + β2
MIRROR 1
MIRROR 2
AUTOCOLLIMATOR
A

Figure 11

$$
\begin{aligned}
r_{A1} &= \Delta_A - \alpha + \beta_1 + \varepsilon_1, \\
r_{A2} &= \Delta_A + \alpha - \beta_2 + \varepsilon_2, \\
r_{B2} &= \Delta_B - \alpha + \beta_3 + \varepsilon_3, \text{ and} \\
r_{B1} &= \Delta_B + \alpha - \beta_4 + \varepsilon_4
\end{aligned}
\tag{4-1}
$$

where Δ_A and Δ_B are the respective autocollimator readings for light which returns parallel to the exit beam, and the ε's are independent error values from a distribution with mean zero and variance σ_1^2. The autocollimators are assumed to be of the usual type where a clockwise rotation of a mirror gives a larger reading. The subscripts of r denote which autocollimator is being read and which mirror is being used.

The above system of four equations and seven unknowns can be solved for α by applying the restraint

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 \tag{4-2}$$

which follows naturally since the sum of the four angles involving β must be exactly 360°. If the error terms are ignored then

$$\alpha = \frac{1}{4}(-r_{A1} + r_{A2} - r_{B2} + r_{B1}). \tag{4-3}$$

The standard deviation of α cannot be computed from equations (4-1) because there is no redundancy in the system, but by taking several sets of four readings the mean value, $\bar{\alpha}$, and its standard deviation $s_{\bar{\alpha}}$, can be computed. Each set of four readings can be made in a couple of minutes so it is reasonable to take several sets of readings with the pentaprism in both the top up and bottom up positions. Suppose that n sets of readings are taken in each position. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ denote the top up values and $\alpha_1', \alpha_2', \ldots, \alpha_n'$ denote the bottom up values. Then

$\bar{\alpha} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i$ and $\bar{\alpha}' = \frac{1}{n}\sum_{i=1}^{n} \alpha_i'$ where n is the number of sets of readings.

In order to test whether the top up values differ from the bottom up values, a test similar to one given by Natrella [4] is employed as follows:

1) Choose the level of significance of the test. (Assume .01)

2) Look up $t_{.995}$ for 2n - 2 degrees of freedom in table 4.

3) Compute $\bar{\alpha}$ and s, $\bar{\alpha}'$ and s' for the two sets of n measurements. (s and s' are the sample standard deviations of the two sets.)

4) Compute $s_p = \sqrt{\frac{s^2 + s'^2}{2}}$. (s_p is the pooled standard deviation.)

5) Compute $u = t_{.995}\, s_p \sqrt{\frac{2}{n}}$.

6) If $|\alpha - \alpha'| > u$, decide that $\bar{\alpha}$ and $\bar{\alpha}'$ do differ. Check the measuring system and repeat the measurements.

7) If $|\alpha - \alpha'| \leq u$, compute $\bar{\bar{\alpha}} = \frac{\bar{\alpha} + \bar{\alpha}'}{2}$, the grand mean of the measured values of the pentaprism deviation from 90°. Compute $s_{\bar{\bar{\alpha}}} = \frac{s_p}{\sqrt{2n}}$, the standard deviation of $\bar{\bar{\alpha}}$.

The uncertainty of $\bar{\bar{\alpha}}$ can be expressed as the sum of random and systematic components of error. The random component is taken to be the three standard deviation limit $3s_{\bar{\bar{\alpha}}}$. The process is thought to be free of systematic errors with the possible exception of the $\Delta\alpha$ values given in table 1. The estimated limit, E, for this error can be added to the random component to give a total uncertainty of the $\bar{\bar{\alpha}}$ estimate of

$$U = 3s_{\bar{\bar{\alpha}}} + E.$$

4.2.2. Angle between Reflected Beams

When a manual autocollimator is set to read the two reflected beams shown in figure 5, the field of view contains a weak and a strong image as shown in figure 12. (Note the position of the non-reflector.) Although the angle between the two beams is 2ξ, the autocollimator divides the angle in half and the separation is read as ξ. (The vertical separation shown in figure 12 is typical but of no consequence.) The readings take the form

$$\begin{aligned} r_s &= \Delta + \varepsilon_1 \\ r_w &= \Delta + \xi + \varepsilon_2 \end{aligned} \qquad (4\text{-}4)$$

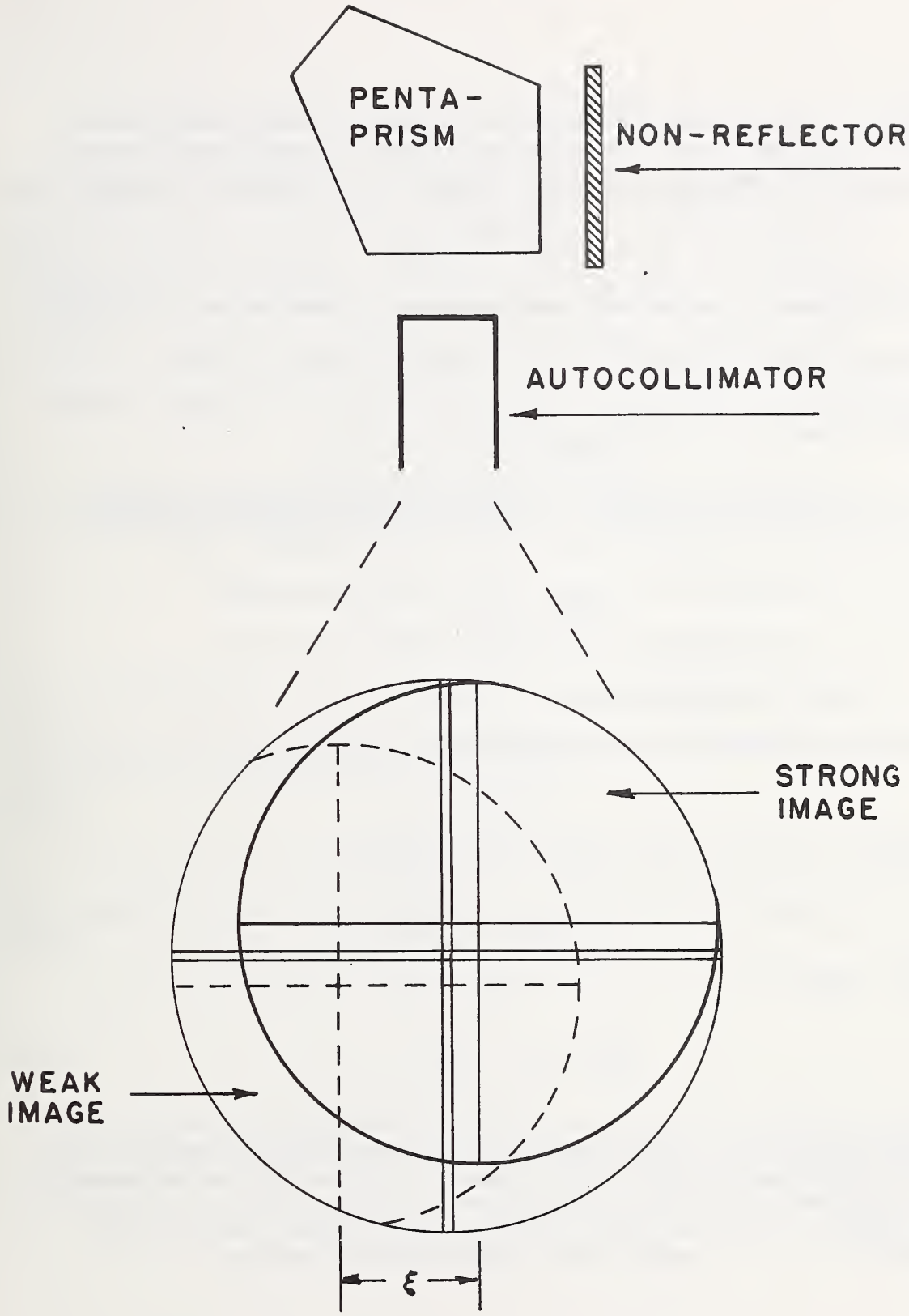

PENTA-
PRISM
NON-REFLECTOR
AUTOCOLLIMATOR
STRONG
IMAGE
WEAK
IMAGE
ξ

Figure 12

where the subscripts of r denote which image is being read. The ε's are independent error values from a distribution with mean zero and variance σ_2^2. Ignoring the error terms as in the previous section, then

$$\xi = r_w - r_s. \tag{4-5}$$

Several pairs of measurements are made and the mean value, $\bar{\xi} = \frac{1}{m}\sum_{j=1}^{m}\xi_j$ and its standard deviation, $s_{\bar{\xi}}$, are computed where m is the number of pairs of readings. (Again the sign of ξ is correct if the autocollimator is of the usual type.)

An explanation should be given for the difference in intensity of the two reflected beams. One beam obviously loses some of its intensity while bouncing around inside the pentaprism. Let

I = intensity of incident beam from autocollimator,

ρ_e = reflectivity of entrance and exit faces, and

ρ_s = reflectivity of silvered faces.

Then the intensity of the strong reflected beam is

$$I_s = \rho_e I, \tag{4-6}$$

and the intensity of the weak reflected beam is

$$I_w = \rho_e(1 - \rho_e)^2\rho_s^4 I\ . \tag{4-7}$$

Their ratio is

$$\frac{I_s}{I_w} = (1 - \rho_e)^2\rho_s^4\ . \tag{4-8}$$

Obviously this ratio may vary considerably depending particularly on the reflectivity of the silvered faces of the given pentaprism. Sometimes the images may be so nearly equal in intensity that one cannot be definitely identified as the weaker. In that case some grease can be applied to the pentaprism face which is 90° away from the autocollimator and the weaker image will gradually disappear.

4.2.3. Indices of Refraction

A perfect pentaprism functions independently of its index of refraction, but an imperfect pentaprism does not. To some degree the index of refraction must be included in the characterization of its performance. It is not a practical matter to measure the index of refraction every time a pentaprism is calibrated, so certain assumptions have to be made. The manufacturer's value for the index of refraction, if available, is taken to be the correct value. Otherwise the value $\mu_2 = 1.5$ is used. This corresponds closely to the "average" value for crown glass. The index of refraction of the air is taken to be $\mu_1 = 1.0$.

4.2.4. Interior Angles

The mean value $\bar{\bar{\alpha}}$ which was computed in section 4.2.1. can be substituted into equation (3-6) to give

$$\bar{\bar{\alpha}} = \frac{\mu_2}{\mu_1}\phi + \left(\frac{\mu_2}{\mu_1} - 1\right)\psi \qquad (4\text{-}9)$$

where the θ_1^2 term is dropped since θ_1 is purposely kept very small. Similarly the mean value $\bar{\xi}$ which was computed in section 4.2.2. can be substituted into equation (3-15) to give

$$\bar{\xi} = \frac{\mu_2}{\mu_1}\phi + \frac{\mu_2}{\mu_1}\psi \quad . \qquad (4\text{-}10)$$

From equations (4-9) and (4-10) the estimates $\hat{\phi}$ and $\hat{\psi}$ are computed to be

$$\hat{\phi} = \left(\frac{\mu_1}{\mu_2} - 1\right)\bar{\xi} + \bar{\bar{\alpha}}$$

and (4-11)

$$\hat{\psi} = \bar{\xi} - \bar{\bar{\alpha}} \quad .$$

Let γ be the deviation from nominal of the 45° angle of the pentaprism (a slight change in notation from section 3.3.). By the reflection properties

$$\gamma = -\frac{1}{2}\phi. \qquad (4\text{-}12)$$

Recall that ψ is the deviation from nominal of the 90° angle of the pentaprism. Then the estimated deviations from nominal of the two interior angles are

$$\hat{\gamma} = \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2}\right)\bar{\xi} - \frac{1}{2}\bar{\bar{\alpha}} \tag{4-13}$$

and

$$\hat{\psi} = \bar{\xi} - \bar{\bar{\alpha}},$$

and their respective standard deviations are

$$s_{\hat{\gamma}} = \sqrt{\frac{1}{4}\left(1 - \frac{\mu_1}{\mu_2}\right)^2 s^2_{\bar{\xi}} + \frac{1}{4}s^2_{\bar{\bar{\alpha}}}}$$

and (4-14)

$$s_{\hat{\psi}} = \sqrt{s^2_{\bar{\xi}} + s^2_{\bar{\bar{\alpha}}}} \quad .$$

Note that if $\bar{\xi} \neq 0$ then $\hat{\gamma}$ is a function of indices of refraction μ_1 and μ_2.

5. Effects when Indices of Refraction Change

The index of refraction of most materials is a function of both temperature (T) and wavelength of light (λ). At high levels of precision errors can arise when a pentaprism is used at a different wavelength and/or temperature than when it was calibrated. These errors can be accounted for by applying correction formulae which take into account the conditions under which the pentaprism was calibrated and the conditions under which it will be used at a given time. Let μ_1 and μ_2 be the indices of refraction during calibration and let μ_1' and μ_2' be the indices of refraction during usage at a later date under different conditions. (For example, the pentaprism could be used in a liquid medium, with a laser, or at an extreme temperature.) The deflection angle of the transmitted beam, α', under the new conditions is then given by

$$\alpha' = \left(\frac{\mu_2'\mu_1}{\mu_1'\mu_2} - 1\right)\bar{\xi} + \bar{\bar{\alpha}}, \tag{5-1}$$

and its standard deviation is given by

$$s_{\alpha'} = \sqrt{\left(\frac{\mu_2'\mu_1}{\mu_1'\mu_2} - 1\right)^2 s_{\bar{\xi}}^2 + s_{\bar{\bar{\alpha}}}^2} \quad . \tag{5-2}$$

The upper bound on the error associated with α' due to a non-zero incident beam angle is given by

$$\Delta\alpha' = \frac{1}{2}\frac{\mu_1}{\mu_2}\left(\frac{\mu_2'}{\mu_1'} - \frac{\mu_1'}{\mu_2'}\right)\bar{\xi}\ \theta_1^2 \tag{5-3}$$

where θ_1 is the largest value of the incident beam's deviation from normal.

In order for the user to make these corrections (if he so chooses), the values T, λ, μ_1, μ_2, $\bar{\bar{\alpha}}$, $\bar{\xi}$, $s_{\bar{\bar{\alpha}}}$, and $s_{\bar{\xi}}$ are all given in the Report of Calibration. The values $\hat{\gamma}$, $\hat{\psi}$, $s_{\hat{\gamma}}$, and $s_{\hat{\psi}}$ are also given.

6. Example

The data sheets for a sample pentaprism calibration are shown in figures 13 and 14. Four sets of measurements of α were made in both the top up and bottom up positions. The statistical test shows that there is no significant difference between the two mean values. Four sets of measurements of ξ were made in the top up position. The index of refraction values were estimated and the remaining parameters computed.

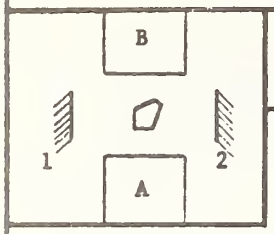
Suppose that this pentaprism were later used in the following two ways.

Case I: Pentaprism used with a laser for which the index of refraction was 1% higher than during calibration.

Case II: Pentaprism used in an underwater test where $\mu_1 = 1.33$.

In both cases assume that the maximum incident beam deviation from normal (θ_1) was 5°, and that $\mu_1 = 1.0$ and $\mu_2 = 1.5$. Using the methods of section 5 the following parameters can be computed.

CALIBRATION OF A PENTAPRISM (OPTICAL SQUARE)



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Serial # T. 472301 (NBS)

Observer CPR & RCV
Date 5/20/75

$a, a' = \frac{1}{4}(-r_{A1} + r_{A2} - r_{B2} + r_{B1})$

		r_{A1}	r_{A2}	r_{B2}	r_{B1}	top up (a)	bottom up (a')
1	top up	7.0	7.5	8.8	2.3	-1.500	
	bottom up	6.9	7.2	8.7	2.4		-1.500
2	top up	7.0	7.5	8.6	2.2	-1.475	
	bottom up	7.0	7.3	8.4	2.4		-1.425
3	top up	6.7	7.4	8.5	2.3	-1.375	
	bottom up	7.0	7.3	8.5	2.2		-1.500
4	top up	7.1	7.3	8.2	2.6	-1.350	
	bottom up	7.1	7.3	8.3	2.3		-1.450
5	top up						
	bottom up						
6	top up						
	bottom up						
7	top up						
	bottom up						
8	top up						
	bottom up						

sets (n) = 4

n	k(n)
2	9.925
3	4.604
4	3.707
5	3.355
6	3.169
7	3.055
8	2.977

mean ($\bar{a}$ and $\bar{a}'$) = -1.425 -1.469

std. dev. (s and s') = .0736 .0375

$s_p = \sqrt{\frac{s^2 + s'^2}{2}} = .0584$

$k(n) = 3.707$

$u = k(n) * s_p * \sqrt{\frac{2}{n}} = .154$ (a)

$|\bar{a} - \bar{a}'| = .044$ (b)

If (b) > (a), check measuring system and repeat measurements.

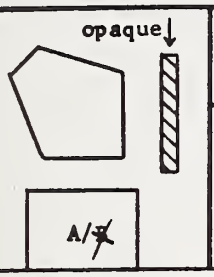
If (b) ≤ (a), compute $\bar{\bar{a}} = \frac{\bar{a} + \bar{a}'}{2} = -1.45$, and $s_{\bar{\bar{a}}} = \frac{s_p}{\sqrt{2n}} = .021$.

Figure 13

5/75

CALIBRATION OF A PENTAPRISM (OPTICAL SQUARE)

sheet 2 of 2



Lot # L 9543
Serial # I.472301

Observer CPR & RCV
Date 5/20/75

Estimated indices of refraction
μ_1 = 1.0
μ_2 = 1.5

Temperature (T) = 20.0 °C.
Estimated wavelength of light (λ) = _____ Å

	image	reading	weak - strong (ξ)
1	weak	5 ' 28.3 "	
	strong	2 ' 48.4 "	2 ' 39.9 "
2	weak	5 ' 28.5 "	
	strong	2 ' 48.4 "	2 ' 40.1 "
3	weak	5 ' 28.6 "	
	strong	2 ' 50.1 "	2 ' 38.5 "
4	weak	5 ' 28.6 "	
	strong	2 ' 48.5 "	2 ' 40.1 "
5	weak	' "	
	strong	' "	' "
6	weak	' "	
	strong	' "	' "

sets (m) = 4

mean ($\bar{\xi}$) = 2 ' 39.7 "

std. dev. (s) = .772 "

$s_{\bar{\xi}} = \frac{s}{\sqrt{m}}$ = .39 "

$\hat{\phi} = (\frac{\mu_1}{\mu_2} - 1)\bar{\xi} + \bar{\alpha}$ = -0 ' -54.6 "

$\hat{\gamma} = -\frac{1}{2}\hat{\phi}$ = 0 ' 27.3 "

$\hat{\psi} = \bar{\xi} - \bar{\alpha}$ = 2 ' 41.1 "

$s_{\hat{\gamma}} = \sqrt{\frac{1}{4}(1 - \frac{\mu_1}{\mu_2})^2 s^2_{\bar{\xi}} + \frac{1}{4}s^2_{\bar{\alpha}}}$ = .07 "

$s_{\hat{\psi}} = \sqrt{s^2_{\bar{\xi}} + s^2_{\bar{\alpha}}}$ = .39 "

Figure 14

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Case I	Case II
$\mu_1' = 1.0$	$\mu_1' = 1.33$
$\mu_2' = 1.515$	$\mu_2' = 1.5$
$\alpha' = 0.2''$	$\alpha' = -41.0''$
$s_{\alpha'} = 0.02''$	$s_{\alpha'} = 0.10''$
$\Delta\alpha' = 0.3''$	$\Delta\alpha' = 0.1''$

7. Conclusion

A discussion of some facts relevant to the calibration of a pentaprism and a description of the calibration process itself have been presented. Three major conclusions can be drawn:

1) Angular errors between the faces of the pentaprism directly affect the angle of deflection of the transmitted beam. If these errors are 5" or less then variations in the angle of incidence (up to 15°) have a negligible effect (less than 0.5") on the angle of deflection of the beam (sec. 3.1.).

2) Angular errors in the vertical alignment of the pentaprism face with the incident beam directly affect the angle of deflection of the incident beam in the vertical plane, but if these errors are less than 3° then they have a negligible effect (less than 0.3") on the deflection of the incident beam in the horizontal plane (sec. 3.3.).

3) The new calibration process allows a thorough characterization of the pentaprism which was lacking in the old calibrations.

Table 1.

Values (in seconds) of $\Delta\alpha = \alpha(\phi, \psi, \theta_1, \mu_1, \mu_2) - \alpha(\phi, \psi, 0, \mu_1, \mu_2)$

$\mu_1 = 1.0$ $\quad\Delta\alpha$ - from exact equation (3-5)

$\mu_2 = 1.5$ $\quad(\Delta\alpha)$ - from approximate equation (3-6)

θ_1	$\phi + \psi$ = 1"	5"	20"	1'	5'	20'
1°	.00	.00	.00	.01	.04	.23
	(.00)	(.00)	(.00)	(.01)	(.04)	(.15)
2°	.00	.00	.01	.03	.16	.76
	(.00)	(.00)	(.01)	(.03)	(.15)	(.61)
3°	.00	.01	.02	.07	.36	1.60
	(.00)	(.01)	(.02)	(.07)	(.34)	(1.37)
5°	.00	.02	.06	.19	.98	4.21
	(.00)	(.02)	(.06)	(.19)	(.95)	(3.81)
7°	.01	.03	.13	.38	1.92	8.07
	(.01)	(.03)	(.12)	(.37)	(1.87)	(7.46)
10°	.01	.07	.26	.78	3.92	16.27
	(.01)	(.06)	(.25)	(.76)	(3.81)	(15.23)
15°	.03	.15	.59	1.78	8.97	36.81
	(.03)	(.14)	(.57)	(1.71)	(8.57)	(34.27)

Table 2.

Observed changes in angle of beam deflection in horizontal and vertical planes vs. angle of pentaprism tilt toward beam.

τ_i = tilt angle of pentaprism on sine bar

h_i = observed angle in horizontal plane

v_i = observed angle in vertical plane

i	τ_i	h_i	$h_i - h_0$	v_i	$v_i - v_0$	$(v_i - v_0) - \tau_i$
0	0' 0.0"	4' 3.2"	0' 0.0"	0' 9.6"	0' 0.0"	0.0"
1	2' 3.8"	4' 2.8"	0' -0.4"	2' 13.5"	2' 3.9"	0.1"
2	4' 7.5"	4' 3.4"	0' 0.2"	4' 15.0"	4' 5.4"	-2.1"
3	6' 11.3"	4' 3.2"	0' 0.0"	6' 21.9"	6' 12.3"	1.0"
4	8' 15.0"	4' 3.5"	0' 0.3"	8' 26.0"	8' 16.4"	1.4"

Table 3.

Predicted changes in angle of beam deflection in horizontal plane vs. angle of pentaprism tilt toward beam, where $\mu_1 = 1.0$ and $\mu_2 = 1.5$.

τ = tilt angle of pentaprism

α = (angle of beam deflection) - 90°

τ	exact α eq. (3-23)	approx. α eq. (3-24)
10'	.0000"	.0000"
30'	.0002"	.0002"
1°	.0036"	.0036"
3°	.2909"	.2907"
5°	2.249"	2.243"
10°	36.25"	35.89"
20°	9' 58.17"	9' 34.19"
30°	53' 9.79"	48' 26.84"

Table 4.

Values of the t distribution
for .01 level of significance

df	$t_{.995}$	df	$t_{.995}$	df	$t_{.995}$
2	9.925	12	3.055	22	2.819
4	4.604	14	2.977	24	2.797
6	3.707	16	2.921	26	2.779
8	3.355	18	2.878	28	2.763
10	3.169	20	2.845	30	2.750

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NBS-114A (REV. 7-73)

U.S. DEPT. OF COMM. BIBLIOGRAPHIC DATA SHEET	1. PUBLICATION OR REPORT NO. NBSIR 76-993	2. Gov't Accession No.	3. Recipient's Accession No.

4. TITLE AND SUBTITLE

The Calibration of a Pentaprism

5. Publication Date January 1976

6. Performing Organization Code

7. AUTHOR(S) Charles P. Reeve
Ralph C. Veale

8. Performing Organ. Report No. NBSIR 76-993

9. PERFORMING ORGANIZATION NAME AND ADDRESS

NATIONAL BUREAU OF STANDARDS
DEPARTMENT OF COMMERCE
WASHINGTON, D.C. 20234

10. Project/Task/Work Unit No. 2131120

11. Contract/Grant No.

12. Sponsoring Organization Name and Complete Address *(Street, City, State, ZIP)*

Same as No. 9

13. Type of Report & Period Covered Final

14. Sponsoring Agency Code

15. SUPPLEMENTARY NOTES

16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

A pentaprism, or optical square, is one of a class of objects known as "constant deviation prisms" whose purpose is to bend a beam of light exactly 90° regardless of the angle of incidence of the light. It is used in many optical measurements where the line of sight or a beam of light must be turned at a right angle. The calibration process which is described is fairly simple and requires two mirrors, two autocollimators, and a surface plate. Consideration is given to the effects of small angular errors in both the internal geometry of the pentaprism and the positioning of the pentaprism relative to an autocollimator. An example of the calibration of the important pentaprism parameters is given. A simple statistical test is employed for testing whether the measurement system is in control.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)*

Autocollimator; calibration; constant deviation prism; geometrical model; optical square; pentaprism; reflection; refraction; standard deviation; uncertainty.

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[ ] Order From National Technical Information Service (NTIS) Springfield, Virginia 22151

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20. SECURITY CLASS (THIS PAGE) UNCLASSIFIED

21. NO. OF PAGES

22. Price

USCOMM-DC 29042-P74